Math 101 Fall 2000 Final Exam **Solutions**

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Wednesday, December 13, 2000

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **three hours**. Do all 12 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ───────────────────────────

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ────── /15

2. ────── /25

3. ────── /20

4. ────── /10

5. ────── /10

6. ────── /15

7. ────── /15

8. ────── /10

9. ────── /20

10. ────── /15

11. ────── /20

12. ────── /25

1. [15 points] Find the following limits, if they exist.

   (a) $\lim\limits_{x \to -1} \frac{x^2-x-2}{x^3+1}$

   This limit is indeterminate of type 0/0 so L'Hôpital gives

   $$\lim_{x \to -1} \frac{x^2 - x - 2}{x^3 + 1} = \lim_{x \to -1} \frac{2x - 1}{3x^2} = \frac{-3}{3} = -1.$$

   (b) $\lim\limits_{x \to 0} \frac{1-\cos 3x}{x \sin x}$

   This limit is indeterminate of type 0/0, so L'Hôpital (twice) gives

   $$\lim_{x \to 0} \frac{1 - \cos 3x}{x \sin x} = \lim_{x \to 0} \frac{3 \sin 3x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{9 \cos 3x}{2 \cos x - x \sin x} = \frac{9}{2}.$$

   (c) $\lim\limits_{x \to 0} (1 + 2x)^{3/x}$

   This limit is indeterminate of type 1^∞ so rearranging and applying L'Hôpital gives

   $$\lim_{x \to 0} (1 + 2x)^{3/x} = \lim_{x \to 0} \exp\left(\frac{3 \ln(1 + 2x)}{x}\right) = \exp\left(\lim_{x \to 0} \frac{3 \ln(1 + 2x)}{x}\right)$$

   $$= \exp\left(\lim_{x \to 0} \frac{3 \frac{2}{1+2x}}{1}\right) = \exp\left(\frac{3\frac{2}{1}}{1}\right) = \exp(6) = e^6.$$

2. [25 points] Find the derivatives of the following functions.

(a) $g(t) = t^2 \sec t$

$$g'(t) = 2t \sec t + t^2 \sec t \tan t.$$

(b) $k(t) = \frac{e^t}{\ln t}$

$$k'(t) = \frac{\ln t \frac{d(e^t)}{dt} - e^t \frac{d(\ln t)}{dt}}{(\ln t)^2} = \frac{e^t \ln t - \frac{e^t}{t}}{(\ln t)^2} = \frac{e^t(t \ln t - 1)}{t(\ln t)^2}.$$

(c) $H(w) = \sqrt{\sin(2w)}$

$$H'(w) = \frac{1}{2\sqrt{\sin(2w)}} \frac{d(\sin(2w))}{dw} = \frac{2\cos(2w)}{2\sqrt{\sin(2w)}} = \frac{\cos(2w)}{\sqrt{\sin(2w)}}.$$

(d) $h(x) = \arcsin(2x - 1)$

$$h'(x) = \frac{1}{\sqrt{1 - (2x - 1)^2}} \frac{d(2x - 1)}{dx} = \frac{2}{\sqrt{1 - (2x - 1)^2}} = \frac{1}{\sqrt{x - x^2}}.$$

(e) $f(x) = \frac{(x-1)^{3/2}(x+2)}{(2x+1)\sqrt{x^2-1}}$

Note that

$$f(x) = \exp\left(\frac{3}{2}\ln(x - 1) + \ln(x + 2) - \ln(2x + 1) - \frac{1}{2}\ln(x^2 - 1)\right).$$

Since $\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$ we have

$$f'(x) = f(x)\frac{d}{dx}\left(\frac{3}{2}\ln(x - 1) + \ln(x + 2) - \ln(2x + 1) - \frac{1}{2}\ln(x^2 - 1)\right)$$

$$= f(x)\left(\frac{3}{2(x - 1)} + \frac{1}{x + 2} - \frac{2}{2x + 1} - \frac{x}{x^2 - 1}\right)$$

$$= \frac{(x - 1)^{3/2}(x + 2)}{(2x + 1)\sqrt{x^2 - 1}}\left(\frac{3}{2(x - 1)} + \frac{1}{x + 2} - \frac{2}{2x + 1} - \frac{x}{x^2 - 1}\right).$$

3. [20 points] Evaluate the following intergals.

(a) $\int \left(1 - 2\sqrt{x}\right)^2 dx$

$$\int \left(1 - 2\sqrt{x}\right)^2 dx = \int \left(1 - 4\sqrt{x} + 4x\right) dx = x - \frac{8}{3}x^{3/2} + 2x^2 + C.$$

(b) $\int e^t \cos(3e^t) dt$

Substituting $u = 3e^t$, $du = 3e^t dt$ gives

$$\int e^t \cos(3e^t) dt = \frac{1}{3} \int \cos(u) du = \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin\left(3e^t\right) + C.$$

(c) $\int \frac{\tan(\sqrt{x})}{\sqrt{x}} dx$

Substituting $u = \sqrt{x}$ (or $x = u^2$) gives $du = \frac{dx}{2\sqrt{x}}$ and

$$\int \frac{\tan(\sqrt{x})}{\sqrt{x}} dx = 2 \int \tan u \, du = -2 \ln |\cos u| + C = -2 \ln \left|\cos(\sqrt{x})\right| + C.$$

(d) $\int \frac{x^2}{1+x^6} dx$

Substituting $u = x^3$, $du = 3x^2 dx$ gives

$$\int \frac{x^2}{1 + x^6} dx = \frac{1}{3} \int \frac{du}{1 + u^2} = \frac{1}{3} \arctan(u) + C = \frac{1}{3} \arctan\left(x^3\right) + C.$$

4. [10 points] Find the equation of the tangent line to the curve $y = \sqrt{4x+1}$ at the point $(2,3)$.

Since

$$\frac{d}{dx}\sqrt{4x+1} = \frac{4}{2\sqrt{4x+1}} = \frac{2}{\sqrt{4x+1}},$$

the slope of the tangent line is

$$m = \frac{d}{dx}\sqrt{4x+1}\bigg|_{x=2} = \frac{2}{\sqrt{9}} = \frac{2}{3}.$$

Since the line goes through $(2,3)$ it is $(y-3) = \frac{2}{3}(x-2)$ or $y = \frac{2}{3}x + \frac{5}{3}$.

5. [10 points] Solve the following initial value problem

$$\frac{dy}{dx} = 4\sin(2x); \ y(0) = 4.$$

One antiderivative of $y'(x) = 4\sin(2x)$ is $-2\cos(2x)$, hence

$$y(x) = C - 2\cos(2x)$$

for some constant C. Plugging in $x = 0$ gives $4 = C - 2$ or $C = 6$. Hence

$$y(x) = 6 - 2\cos(2x).$$

6. [15 points] Find the area of the region in the plane bounded by the curves $y = x(x-3)^2$ and $y = x$. (Be careful, there are two pieces to this region.)

The two curves intersect when $x = x(x-3)^2$ or $0 = x^3 - 6x^2 + 8x = x(x-2)(x-4)$, hence at $x = 0$, 2, or 4. Since there are three intersection points there will be two regions. At $x = 1$, the cubic is $y = 4$ and the line is $y = 1$, hence the cubic is higher between $x = 0$ and $x = 2$. At $x = 3$, the cubic is $y = 0$ and the line is $y = 3$. Hence the line is higher between $x = 2$ and $x = 4$. Thus the total area is

$$A = \int_0^2 \left(x(x-3)^2 - x\right)\, dx + \int_2^4 \left(x - x(x-3)^2\right)\, dx.$$

Multiplying out and integrating gives

$$A = \int_0^2 \left(x^3 - 6x^2 + 8x\right)\, dx + \int_2^4 \left(-x^3 + 6x^2 - 8x\right)\, dx$$
$$= \left(\frac{1}{4}x^4 - 2x^3 + 4x^2\right)\Big|_{x=0}^{2} + \left(-\frac{1}{4}x^4 + 2x^3 - 4x^2\right)\Big|_{x=2}^{4}$$
$$= (4 - 16 + 16) - 0 + (-64 + 128 - 64) - (-4 + 16 - 16) = 8.$$

7. [15 points] Find the length of the curve $y = x^{3/2} - \frac{1}{3}x^{1/2}$ from $x = 1$ to $x = 4$.

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2}$$

hence

$$(f'(x))^2 = \frac{9}{4}x - \frac{1}{2} + \frac{1}{36x}.$$

Thus

$$1 + (f'(x))^2 = \frac{9}{4}x + \frac{1}{2} + \frac{1}{36x} = \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right)^2.$$

Hence

$$L = \int_1^4 \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right) dx = \left(x^{3/2} + \frac{1}{3}x^{1/2}\right)\Big|_1^4$$

$$= \left(8 + \frac{2}{3}\right) - \left(1 + \frac{1}{3}\right) = \frac{22}{3}.$$

8. [10 points] Let S be the surface that results from revolving the curve $x = \tan y$ from $y = 0$ to $y = \pi/3$ about the x-axis. Express the surface area of S as a definite integral, but do not attempt to evaluate the integral.

Let $x = g(y) = \tan y$. Then

$$g'(y) = \sec^2 y, \text{ and } 1 + (g'(y))^2 = 1 + \sec^4 y.$$

Since the curve is revolved about the x-axis, the radius will be y. Hence

$$S = \int_0^{\pi/3} 2\pi y \sqrt{1 + \sec^4 y}\, dy.$$

9. [20 points] Let R be the region in the plane bounded by the curve $y = \sqrt[3]{1+x}$, the line $x = 7$, and the line $y = 1$. Let S be the solid that results from revolving R about the x-axis. Express the volume of S as an integral in TWO ways, using the method of disks/washers and using the method of shells, and evaluate ONE of the two integrals (your choice).

Since the region is revolved about the x-axis, to use the method of washers we will need to do an integral with respect to x. The lowest x value is $x = 0$ when the curves $y = \sqrt[3]{1+x}$ and $y = 1$ meet. The highest x value is $x = 7$, at the vertical line. For $0 \le x \le 7$, the region is below $y = \sqrt[3]{1+x} = f(x)$ and above $y = 1 = g(x)$. Hence the volume is

$$V = \int_0^7 \pi \left((1+x)^{2/3} - 1^2 \right) dx = \pi \left(\frac{3}{5}(1+x)^{5/3} - x \right) \Big|_0^7$$

$$= \pi \left(\frac{3}{5}8^{5/3} - 7 - \frac{3}{5}1^{5/3} + 0 \right) = \pi \left(\frac{96}{5} - 7 - \frac{3}{5} \right) = \frac{58\pi}{5}.$$

To use the method of shells we must do an integral with respect to y. The highest y value that occurs is $y = 2$ when the curve $y = \sqrt[3]{1+x}$ meets the line $x = 7$, the lowest y value is $y = 1$ at the horizontal line. For $1 \le y \le 2$, the right side of the region is the line $x = 7$ and the left side is the curve $y = \sqrt[3]{1+x}$ or $x = y^3 - 1$. Hence the volume is

$$V = \int_1^2 2\pi y \left(7 - (y^3 - 1) \right) dy = 2\pi \int_1^2 \left(8y - y^4 \right) dy$$

$$= 2\pi \left(4y^2 - \frac{1}{5}y^5 \right) \Big|_1^2 = 2\pi \left(\left(16 - \frac{32}{5} \right) - \left(4 - \frac{1}{5} \right) \right)$$

$$= \frac{58\pi}{5}.$$

10. [15 points] A bacteria culture contains 1000 bacteria 3 hours after it is started and 3000 bacteria 5 hours after it is started. Assuming exponential growth, find a formula for the number of bacteria present t hours after the culture was started and use this formula to estimate the initial number of bacteria present and the time when there will be 20,000 bacteria present.

Since the number of bacteria grows exponentially, the number $N(t)$ after time t will be given by $N(t) = N_0 e^{kt}$ for some constants N_0 and k. Plugging in $t = 3$ and $t = 5$ give

$$1000 = N(3) = N_0 e^{3k} \text{ and } 3000 = N(5) = N_0 e^{5k}.$$

Dividing these we see $3 = e^{2k}$ or $2k = \ln 3$ or $k = (\ln 3)/2$. Plugging this in gives the initial number of bacteria as

$$N_0 = 1000 e^{-3k} = 1000 \cdot e^{-3(\ln 3)/2} = \frac{1000}{3\sqrt{3}}.$$

The time t when there are 20000 bacteria present will solve

$$20000 = N(t) = N_0 e^{kt} = \frac{1000}{3\sqrt{3}} e^{t(\ln 3)/2}.$$

This gives $60\sqrt{3} = e^{t(\ln 3)/2}$ or $t(\ln 3)/2 = \ln(60\sqrt{3})$ or

$$t = \frac{2\ln(60\sqrt{3})}{\ln 3}.$$

11. [20 points] (a) We want to make a cylindrical can without a top which has a volume of 1000π in^3. What should the dimensions be in order to minimze the amount of material used to make the can (assuming no material is wasted in the construction of the can).

Let r be the radius of the can and h the height, both in inches. Since we want the volume to be 1000π these are related by

$$1000\pi = \pi r^2 h \text{ or } h = \frac{1000}{r^2}.$$

The curved area of the can will be $2\pi rh$ and the bottom will be πr^2 hence the total area will be

$$A(r) = 2\pi rh + \pi r^2 = \frac{2000\pi}{r} + \pi r^2.$$

Since r and h are lengths we must have $r > 0$, so the domain is $(0, \infty)$. We compute

$$A'(r) = -\frac{2000\pi}{r^2} + 2\pi r.$$

This is defined for $r > 0$ so the only critical point will be when the derivative is zero, that is,

$$\frac{2000\pi}{r^2} = 2\pi r \text{ or } r^3 = 1000 \text{ or } r = 10.$$

The critical point corresponds to dimensions $r = 10$ inches and $h = 10$ inches. To see that this is really a global minimum we use either the first derivative test or the second derivative test. With the first derivative test: Since

$$A'(x) = \frac{2\pi(r^3 - 1000)}{r^2}$$

we see that $A'(x) < 0$ for $0 < r < 10$ and $A'(x) > 0$ for $r > 10$, hence $r = 10$ is a global minimum of $A(r)$. With the second derivative test: We compute

$$A''(r) = \frac{4000\pi}{r^3} + 2\pi.$$

Since this is positive for ALL $r > 0$, $r = 10$ is a global minimum.

(b) Justify in a few lines why your answer to part (a) is really an absolute minimum.

See either the first or second derivative tests described above.

12. [25 points] For the function $f(x) = \frac{x^2-3}{x+2}$, the first two derivatives are $f'(x) = \frac{x^2+4x+3}{(x+2)^2}$ and $f''(x) = \frac{2}{(x+2)^3}$. YOU ARE NOT REQUIRED TO VERIFY THESE FORMULAS. For all other aspects of this problem you are required to justify your answer.

(a) Find all horizontal and vertical asymptotes of the graph $y = f(x)$. Be sure to give the limits you need to show these are asymptotes.

Since
$$\lim_{x \to \infty} \frac{x^2-3}{x+2} = \lim_{x \to \infty} \frac{2x}{1} = \infty, \text{ and}$$
$$\lim_{x \to -\infty} \frac{x^2-3}{x+2} = \lim_{x \to -\infty} \frac{2x}{1} = -\infty$$
there are no horizontal asymptotes. The only possible vertical asymptote is at the discontinuity $x = -2$. As $x \to -2^+$, $x^2 - 3$ is nearly 1 (hence positive) and $x + 2$ is a small positive number so
$$\lim_{x \to -2^+} \frac{x^2-3}{x+2} = +\infty.$$

As $x \to -2^-$, $x^2 - 3$ is still nearly 1 and $x + 2$ is a small negative number so
$$\lim_{x \to -2^-} \frac{x^2-3}{x+2} = -\infty.$$

Thus $x = -2$ is a vertical asymptote.

(b) Find the open intervals on which the function f is increasing and those on which it is decreasing.

Note $f'(x) = \frac{x^2+4x+3}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2}$. Since $x + 1$ is positive for $x > -1$ and negative for $x < -1$, $x + 3$ is positive for $x > -3$ and negative for $x < -3$, and $(x+2)^2$ is positive for $x \neq -2$ we have $f'(x) > 0$ for $x < -3$, $f'(x) < 0$ for $-3 < x < -2$ and for $-2 < x < -1$, and $f'(x) > 0$ for $x > -1$. Hence f is increasing on $(-\infty, -3)$ and on $(-1, \infty)$ and f is decreasing on $(-3, -2)$ and on $(-2, -1)$. It is not true that f is decreasing on $(-3, -1)$ because of the discontinuity at $x = -2$.

(c) Find all critical points of $f(x)$ and determine whether they are local maxima, local minima or neither. Justify your answer.

The only place where $f'(x)$ does not exist is at the asymptote $x = -2$, which does not give a critical point. The places where $f'(x) = 0$ are at $x = -1$ and $x = -3$ so these are the critical points. By the first derivative test and part (b), $x = -3$ is a local maximum and $x = -1$ is a local minimum. (One may also use $f''(-3) = -2$, $f''(-1) = 2$ and the second derivative test.)

(d) Find the intervals on which the function f is concave upward and those on which it is concave downward.

Since $x + 2$ (and hence $f''(x)$) is positive for $x > -2$ and negative for $x < -2$, f is concave upward on $(-2, \infty)$ and is concave downward on $(-\infty, -2)$.

(e) Sketch the graph of $y = \frac{x^2 - 3}{x + 2}$ using your answers to parts (a)-(d) and any additional information required.